RESEARCH FOR INNOVATION:

A TEACHING SEQUENCE ON THE ARGUMENTATIVE APPROACH TO PROBABILISTIC THINKING IN GRADES I-V AND SOME RELATED BASIC RESEARCH RESULTS

Paolo BOERO*, Valeria CONSOGNO*, Elda GUALA*, Teresa GAZZOLO*

(RECH. EN DID. DES MATH., 2009, VOL. 29-1, 59-96) RESUME

Cet article se situe dans le cadre de la « recherche pour l'innovation » italienne ; il présente la préparation, la planification, la réalisation en classe, l'analyse et l'expérimentation plus étendue d'un enseignement innovant sur le long terme destiné aux classes du primaire (de la première à la cinquième année). Cet enseignement concerne l'approche de la pensée probabiliste par l'argumentation ; il est conçu dans la perspective de l'enseignement-apprentissage des mathématiques dans les domaines d'expérience. L'article présente aussi quelques développements de recherches fondamentales concernant les hypothèses sous-jacentes à la planification des séquences d'enseignement et leur réalisation en classe. Ces développements concernent les mécanismes de construction argumentative de savoirs de probabilité en classe : trois mécanismes (liés aux fonctions constructives du langage naturel) seront exposés, et les conditions de leur mise en fonctionnement seront discutées.

Mots-clés : didactique des mathématiques, domaines d'expérience, probabilité, recherche pour l'innovation

RESUMEN

En el marco de la "investigación para la innovación" en Italia, este artículo describe la preparación, planeamiento, realización en clase, revisión, y

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experimentación de un a secuencia didáctica de largo alcance (Grados 1 a 5) concentrada en la aproximación argumentativa al pensamiento probabilístico. Esta secuencia didáctica deriva de nuestra perspectiva experiencial hacia el proceso de enseñanza-aprendizaje. El artículo presenta asimismo desarrollos de nuestra investigación básica, concentrada en las hipótesis subyacentes en el planeamiento de la secuencia didáctica y su realización en la clase. En particular, la investigación básica presentada concierne los mecanismos de construcción argumentativa del conocimiento probabilístico en la clase: tres mecanismos (relacionados con las funciones constructivas del lenguaje natural) se presentan y se discuten las condiciones para su activación.

Palabras-clave: didáctica de las matemáticas, campos de experiencias, probabilidad, investigación para la innovación

ABSTRACT

In the frame of the Italian “research for innovation”, this paper presents the preparation, planning, classroom implementation, revision and wider experimentation of a long term (Grade I - Grade V) innovative teaching sequence on the argumentative approach to probabilistic thinking, conceived in the perspective of the teaching and learning of mathematics in the fields of experience. It also presents some developments of basic research, which deal with the hypotheses underlying the planning of the teaching sequence and its classroom implementation. In particular, the basic research presented concerns the mechanisms of argumentative construction of probability knowledge in the classroom: three mechanisms (related to the constructive functions of natural language) will be exhibited, and the conditions for their activation will be discussed.

Key words: didactics of mathematics, fields of experience, probability, research for innovation
INTRODUCTION

The development of research in mathematics education depends on local constraints, opportunities and traditions. In Italy, national programs and prescriptions for curricula are rather loose: teachers teach the same students for an entire cycle (in primary school, from grade I to grade V); research in mathematics education developed (during the seventies) as a cultural and social engagement of some mathematicians, strongly related to historical and epistemological perspectives; and psychology has been considered (in the last thirty years) as a discipline that can provide tools (of different origin) for research in mathematics education, rather than as a paradigm for research in our field according to the methods and hypotheses of a given school (be it Piaget's constructivism, behaviorism, or other). The same is true for theoretical perspectives elaborated in the field of mathematics education (e.g. Brousseau's Theory of Didactical Situations, or Realistic Mathematics Education of the Freudenthal Institute): we consider ourselves free to use tools coming from those theories without adopting one of them, with only the requirement that tools coming from different theories be locally compatible (for more details about the development of Italian research in Mathematics Education in the last forty years, see Boero & Dapueto, 2007). In the Italian situation, a central role, among the aims of research in mathematics education (cf. Boero & Radnai Szendrei, 1998), is played by the planning and classroom implementation of teaching sequences concerning important subjects (from the perspective of providing teachers with examples of how to improve teaching according to societal needs), as well as the production of knowledge concerning the mechanisms of teaching and learning in school and the conditions that can improve learning results. Italian scholars contributed to the field by establishing a narrow, dialectic relationship between those two aims, synthesized in the expression “Research for innovation” (Arzarello and Bartolini Bussi, 1998). Planning and implementing long-term innovations (based on historical and epistemological reflections on their mathematical content as well as on analyses of current difficulties met by teachers in school) offers opportunities to point out teaching and learning problems inherent in the innovation, which need to be tackled with tools provided by research in psychology, mathematics education and other disciplines. Results concerning those problems can be applied to the improvement of teaching sequences (challenging and refining working hypotheses and methods, interpreting teaching and learning difficulties with the aim of
overcoming them, etc.); further experimental work concerning the teaching sequences can provide elements for further basic research, and so on.

Two typical features of the Italian Research for innovation, inherent in its development, are: the role played by teachers who both act as researchers in the local research teams promoted by university researchers and carry out the crucial teaching experiments in their classrooms (Malara & Zan, 2002), and the elaboration of ad hoc original theoretical tools that are needed to frame and orient long-term classroom activities (for instance, Bartolini’s Mathematical discussion: Bartolini Bussi, 1996; Boero’s Field of experience; Boero et al., 1995; Arzarello’s Space of action, production, and communication, Arzarello, 2008).

According to the main features of Research for innovation, this paper has several goals. First, we will present the planning of the long-term sequence concerning the approach to probabilistic thinking from grade I to grade V (its rooting in epistemological, historical and psychological literature, and the main educational choices concerning its classroom implementation according to the perspective of teaching and learning mathematics in fields of experience), and some data and reflections on its experimentation as a prototype from grade I to grade IV and on its replication in other classes (see Section III). Second, we will consider the working hypothesis underlying the planning of the sequence—namely, the students’ argumentative approach to some crucial aspects of probabilistic thinking—and present some related basic research developments (see Sections IV and V). These developments concern the mechanisms of argumentative social construction of knowledge and the conditions that can foster or prevent the activation of those mechanisms. In this way, we try to show how the teaching of mathematics in fields of experience exemplified in Section III (in particular, see III.2, III.3, III.4 regarding the didactics of fields of experience) works as a long-term educational context for local in-depth analyses and research developments concerning specific teaching and learning problems related to the implementation of this approach.

I. THE NEED FOR AN EARLY APPROACH TO PROBABILISTIC THINKING

In the Italian National Programs issued in 1985, the introduction of elements of probability and statistics in primary school (a subject never before taught at that level in Italy) was set as one of five major
goals of mathematics instruction in primary school. This idea was suggested by curricula implemented at the time in several other countries, and it was also related to the need for an early intervention concerning ways of thinking that are very common in the Italian popular culture. Indeed, many nationally circulated newspapers publish data concerning late numbers of the Lotto game, and many people bet large amounts of money on those numbers; superstitious practices are very common, yet even some important people (political leaders, managers of big enterprises, etc.) use superstitious practices in public.

As is usual for Italian programs, the 1985 prescription of introducing elements of probability and statistics in primary school did not specify many details for the concrete classroom implementation (no indication about what to do in each grade, no list of final competencies, etc.); only some principles and loose methodological indications were provided (e.g. “to move from the correct use of words like possible, impossible, etc. to a more precise measure of the degree of uncertainty of an event”). Due to the fact that most primary school teachers did not know probability and knew only some elementary notions of statistics (e.g. frequency, mean value, etc.), probability and statistics was one of the subjects of the national plan for in-service teacher training, which followed the National Programs issued in 1985. Nowadays, teachers teach probability as a minor subject, following textbooks that dedicate 3-4 pages to it, with some trivial exercises that involve calculating probabilities for stereotyped random situations. Mathematics educators have tried to contribute to a more relevant impact of the National Programs in school by collaborating on the national plan of in-service teacher training and by elaborating, experimenting and disseminating (through reviews for teachers) more advanced teaching sequences on the subject (with the aim of preparing students to deal with more complex and less stereotyped random situations). Also, the Genoa Research Team engaged in this effort for some years. We tried to build and experiment with sequences of tasks aimed at a more deep and conscious approach to the concept of probability and to some concepts and tools of statistics (frequency, mean value, histograms, etc.). The results were rather good as concerns the calculation of probability, frequency, mean value, etc. in situations that were more complex than those proposed in the current textbooks, but poor as concerns the students' capacity to control and overcome some of the most frequent misconceptions (particularly the idea that chance tends to re-equilibrate anomalous peaks of outcomes, or those inherent in the existence of lucky people, or lucky days, or superstitious practices,
etc.), and to distinguish between random events and other events that at least partially depend on human choices. In other words, while the results were rather satisfactory as concerns the purely mathematical side, they were completely unsatisfactory on the cultural side. These unsatisfactory results suggested a need for an entirely new approach to the early teaching of probability and statistics, based on systematic argumentative activity starting in Grade I, related to students' conceptions. The calculation of probability was postponed to Grades IV or V as one of the activities in this field (neither the most important nor the most difficult). For us, this change of perspective involved considering the domain of random phenomena as a Field of experience and not as a mere collection of exercises concerning probability, thus making the work on probability more coherent with the whole Genoa Research Team Project for primary school.

II. THE FIELD OF EXPERIENCE OF RANDOM PHENOMENA

The project developed by the Genoa Research Team was conceived from the perspective of the work in suitable fields of experience. The construct of Field of experience was introduced (Boero, 1989; Boero et al., 1995) and refined (Dapueto & Parenti, 1999; Douek, 2003) in order to deal with the problems met when, in teaching/learning mathematics, we refer to contexts that the student are acquainted with. In short, by Field of experience we mean a sector of human culture that the teacher and students can recognize and consider as unitary and homogeneous (examples are the field of experience of the Sun shadows and that of Purchases and money). In the long run, arithmetic, too, may become a field of experience for students. Two delicate issues concern the delineation of a field of experience and the fact that some fields of experience are recognized as such by the students only in the long run of their classroom activities related to them (Douek, 2003). Concerning the first issue, we can say that the adjective unitary refers to the fact that a field of experience covers a set of phenomena, properties, relationships that are difficult to separate without losing some of its characteristic aspects. For instance, in the Money and purchases field of experience, if we separate money from purchases, one of the crucial functions of money would be lost, and purchases would have no means to occur (at least in our society) without money. The adjective homogeneous refers to stability of frames and scripts (Schank & Abelson, 1977; Tannen, 1979) across the situations of a given field of experience; in particular, the meanings of crucial linguistic expressions related to it do not change from one situation to the other: for instance, in the...
Money and purchases field of experience, the expression to pay a price has the same meaning in all situations of purchase and has no metaphorical meaning. Concerning the second issue, the fact that a field of experience is a sector of human culture allows for fields of experience well-established in the teacher's mind that are only potential entities for the students (like arithmetic or genetics). More generally, we can say that, for almost all fields of experience, the students' initial knowledge and experience about them is only partial but grows during the long-term classroom activities organized by the teacher.

With reference to the previous remarks, we note that when dealing with teaching-learning problems related to a given field of experience, the complex relationships that develop at school between the student's inner context (experience, mental representations, procedures concerning the field of experience), the teacher's inner context, and the external context (signs, objects, objective constraints specific to the field of experience) must be considered.

In previous papers that refer to the construct of field of experience (Douek, 1999; Bartolini Bussi, Boni, Ferri, Garutti, 1999), the authors have considered how the evolution of the student's inner context is helped by activities organized and guided by the teacher within a given real-world field of experience. In real-world fields of experience, the student may acquire mathematical tools and thinking strategies that he/she will use to think and act more effectively within the same or within other fields of experience (as concerns the problem of transfer, see Dapueto & Parenti, 1999). These tools may also become the basic elements to approach (through the teacher's mediation) the mathematical fields of experience.

In this perspective, the problem of the relationship between culture and mathematics teaching/learning plays a key role if we want to understand the contributions that real-world contexts may give to the development of mathematical knowledge and skills, and the contributions that mathematics may give to the cultural mastering of different real-world contexts. In turn, such understanding seems to be necessary to clarify the potential of, limits of, and variables governing the effective use of real-world contexts in teaching mathematics.

Concerning this issue, with an eye to the field of experience of random phenomena, we remark that real-world contexts are generally used in teaching mathematics in order to connect studying mathematics with out-of-school motivations and applications. Such use often takes on ideological and social connotations which may lead to:
i) undervaluing the difficulties sometimes involved when the complexity and difficulty of the subject matter (mathematics) are interlaced with the difficulties regarding the cultural mastering of some real-world contexts (which are, of course, experienced out of school but not so explicitly and rationally as the mathematical modeling process requires);  

ii) not maximizing the potentials of working in real-world contexts to develop skills and attitudes needed for activities within the domain of mathematics;  

iii) disregarding one of the tasks school has to perform – handing over to new generations mathematics as a science relatively independent of its applications – or disregarding (when this task is undertaken) the difference between mathematics considered as a tool (often used in a not fully aware and explicit form) to act in real-world contexts and mathematics considered as a science relatively independent of its applications.

As to i) (cultural mastering of real-world contexts), our previous studies (Boero et al., 1995) suggest that it is useful to distinguish:

- real-world fields of experience that are already mathematised in out-of-school life (such as those usually involving handling money or measurement of lengths, time and weight);
- real-world fields of experience in which the mathematical modelling activity carried out at school may be clashing with conceptions rooted in common sense, or at least it cannot rely on sufficient levels of mathematisation already existing in everyday culture (a good example is the transmission of hereditary characters).

We think that such a distinction is important to clarify how, in the first case, the teacher's work can rely on out-of-school experience to develop concepts and mathematical procedures and to build up higher levels of awareness and explicitness regarding the mathematical tools and processes involved, while, in the second case, this does not happen and sometimes the teacher has to work against conceptions opposed to mathematical modelling.

As to ii) (potentials of activities related to real-world contexts), previous papers (Douek, 1999; Bartolini Bussi et al., 1999) have tried to demonstrate how some basic skills and processes involved in mathematical activities (such as linguistic-reasoning skills, meta-cognitive processes, etc.) can be developed by utilizing the potentials of working in real-world experience fields.

As to iii) (relationship between mathematics as a tool to act in real-world contexts and mathematics as an independent science), previous papers (Boero et al., 1995; Dapueto & Parenti, 1999) have suggested
that, on the one hand, there is no gap between some skills and some concepts which can be built up by working in real-world fields of experience and used for working in mathematics; but, on the other hand, there are gaps between everyday thinking about reality, and thinking about reality through mathematical models, as well as between the rules of everyday thinking, and the rules of mathematical thinking. We believe that teaching mathematics must incorporate all this (continuity as well as discontinuity).

In what sense can the sequence concerning the approach to probability thinking from Grade I to Grade IV be situated in the perspective of fields of experience? The sequence is not focused on a phenomenon from everyday life (like in the cases of sun shadows or buying and selling goods), or on a mathematical subject that progressively becomes a familiar subject for students, resonant with culture (like in the case of elementary arithmetic). The sequence is focused on a way of thinking about a wide set of phenomena, one part of which belongs to students’ everyday life experience: they know that adults play random games, invest money to win the lotto, etc. Considering random phenomena, we can still speak of a field of experience in the sense of a domain of phenomena that can be progressively identified by students as related to the same rationale. In this sense, the field of experience is progressively built in the classroom as a relatively autonomous cultural domain that should not be detached (in primary school) from related everyday life phenomena and that represents a cultural frame for them. At the beginning of the sequence, random situations are related to the ordinary life of the classroom (particularly random choice of the students when performing specific activities) or to other fields of experience (e.g. the time of the calendar: weather observations and forecasting, season changes, etc.). Over time, the choice of the tasks extends to random games (coins, dice, raffle, etc.) that create resonances with out-of-school adult experiences.

From the perspective of the teaching and learning of mathematics in the fields of experience, we can say that the transition from our previous experiments on the introduction of probability in school as calculation of probability for standard, stereotyped random situations to the present argumentative approach to probability thinking can be considered as a transition from a mathematics-oriented approach focused on the solution of mathematical tasks, to a cultural approach. From the same perspective, we can say that the change is rather similar to the transition from teaching geometry or numbers as mathematical subjects, to teaching geometry as a way of modelling
space phenomena, or numbers as tools to solve everyday life problems (measuring lengths, buying goods, evaluating time intervals, etc.).

III. THE PLANNING AND CLASSROOM IMPLEMENTATION OF THE TEACHING SEQUENCE

The sequence of didactical situations from Grade I to Grade IV was designed around steps that were considered relevant in the individual and/or historical development of probabilistic thinking (Piaget, 1951; Fischbein et al., 1971; Fischbein and Gazit, 1984; Hacking, 1975).

III. 1. Sequencing the approach to probabilistic thinking

The selected steps have been considered as objects of specific tasks designed to promote the students’ direct encounter with the inherent cultural advances (see later: Episodes 1 for step 5, Episode 2 for step 6, Episode 3 for step 8). Steps can consist of one or more tasks, and some steps can cover a long period (many months): they consist of several tasks and can be performed in parallel with other steps; thus, steps do not form a linear progression. The number of tasks was 22 (for about 90 hours of classroom work) in the first five experimental classes. Note that the entire classroom time spent on the sequence (about 10% of the time usually devoted to mathematics in the five grades of primary school) covers one of the main content topics of the present Italian curricular indications, together with one part of the indications concerning the development of mathematical argumentation in primary school.

S1 (grades I - II): moving from egocentric estimations of what is likely to happen in the future (based on the child’s wish) or the idea of a finalistic determination of what is going to happen, to a more objective consideration of the random nature of events.

Mediation tools are introduced: tables, histograms. These work as comparison and memory tools and allow students to make their perception of random phenomena more objective (Fischbein & Gazit, 1984), and they will be systematically used for the next steps (when appropriate). Two tasks are devoted to this step, and they concern: predicting the weather for the day of an excursion the next week, given the data of the last twenty years; predicting the next outcomes of the drawings for classroom tasks that some students like and other students dislike, given the past outcomes. As usual, students must explain their answers.
S2 (grades II - III): discussing the idea of a lucky person or a lucky number.

This idea (related to the previous ones in a rather sophisticated way: Piaget, 1951; Fischbein et al., 1971; Hacking, 1975) resists the previous discussions and needs further classroom work based on the comparison of several random situations and individual students’ chances in those situations. One task is devoted to this step: it concerns a raffle (with real prizes), where students bet on a couple of numbers and then interpret what happened. The task can be given more than one time (in the case of the first experimental class, 4 times over one year), with different random games.

S3 (grades II-III): calling into question the idea that a person can influence the outcomes of a random phenomenon according to his/her will and superstitious practices (Piaget, 1951; Hacking, 1975).

Two tasks are devoted to this step; students are asked to write whether and how they can influence the outcome of the drawing for a new task (that all students dislike). Once several drawings have been made, students are asked to write what they think about the results.

S4 (grades II-III): using data of drawings (with replacement) to guess the composition of a box.

This step is very important from the epistemological point of view for three different reasons: because it introduces the idea that a growing number of outcomes for a purely random phenomenon can provide more and more reliable information for a reasonable prediction; because it shows how irregular the sequence of the outcomes can be; and because it is the first encounter with the fact that a box with 2 black counters and 3 red counters is not distinguishable from a box with (say) 4 black counters and 6 red counters, if we consider series of drawings with replacement (thus, the first encounter with the concept of ratio, even if the teacher does not yet focus on it). The task (which should be repeated at least three times, the first one with equal number of counters of the two colors, the second with different numbers, the third with a third rare color) consists in predicting the composition after 20, 50, 100, 200 drawings, and writing comments about the reliability of the prediction.
S5 (grade III): the explicit approach to the notion of probability of an event as the ratio between the number of favourable outcomes and the number of all possible equally likely outcomes.

In our project, the notion of ratio has not yet been introduced in explicit terms, so (after the informal encounter in the previous situation) students focus on it for the first time with this reference situation (Vergnaud, 1990). In the early history of probability, this construction meant the separation between the superstitious or fatalistic view of random events, and their evaluation in terms of objective measures of probability (Hacking, 1975). Two similar tasks are related to this step. They concern the comparison between two different games, in order to choose the game where it is easier to win. The first task involves comparing the coin game (to bet on one side of the coin) and the dice game (to bet on one of the six digits); the second task involves comparing two raffles with different quantities of numbers.

S6 (grade III): creating necessity for the hypothesis of equally likely outcomes for a fair determination of the more likely event.

In the first task, students are asked to decide whether it is more convenient to bet on an even or odd result of the sum of the numbers obtained by casting two dice. Students should learn to distinguish between a complex outcome (e.g. 4 as the result of the sum of the numbers obtained by casting two dice) and the elementary equally likely outcomes that comprise it (e.g. 4 as 1+3 or 2+2 or 3+1), thus beginning to develop the combinatorial approach to probability. According to Hacking (1975), this was a crucial progress in the early history of probability: Galileo struggled with it when trying to determine whether the number that results if one casts three dice and adds their digits is more likely to be between 9 and 10 or between 11 and 12. The second task concerns the Chinese morra game (two people bet on odd or even, then they raise their hands, each with a number from 0 to 5 of fingers out, and sum those two numbers). Students are asked to decide if it is better to bet on odd or even. Students are expected to apply the same criteria of the previous game. The provisional conclusion concerning this analogy will offer the possibility of a comparison between the two games (S8), which will show their differences.
S7 (grades III and IV): “chance has no memory” and stability of frequency close to the expected value of probability for a large number of outcomes.

This step is connected to S4, but now the focus is on the regularity of outcomes of the heads or tails game. In the first task, students are asked to predict what will happen after four consecutive heads; in the second task, they are asked to write their comments on a long series of outcomes, where there are several series of four consecutive heads; in the third task, they are asked to write their comments (and explanations) concerning the fact that the frequency of heads in the same series of outcomes is rather near to 0.5 after 2000 outcomes. During the third task, the Cartesian graphic representation of the frequency/outcomes relation is introduced.

S8 (grade IV): comparison between purely random situations and situations where human decisions and habits may influence outcomes.

In our sequence, the games with the two dice and the Chinese morra are compared. One task is related to this step. Students are asked to compare the histograms concerning a high number (500) of outcomes for both games, and to interpret the most relevant differences. Histograms are very different because the highest value, 10, in the Chinese morra game is much more likely to happen (due to a spontaneous human tendency) than the highest value, 12, in the two dice game; the opposite happens for 0 and 2.

S9 (grade IV): evaluating the probability of complex random rough phenomena that need modelling.

This step consists in three tasks: given four counters with the letters A, O, R, M,

- to evaluate the probability to get a meaningful word in Italian (after four drawings without replacement);
- to establish if it is better to draw 3 names of students all together, or 1 name each time for three times, in order to win a prize (drawings with replacement);
- in the raffle (4 numbers for each student), to decide if it is better or not to change the personal set of four numbers with the set of numbers offered by the teacher to each student, after two drawings.

III. 2. The teaching and learning hypotheses

The field of experience approach to probabilistic thinking means an intentional intervention on students’ ideas (either derived from their cultural environment or spontaneous constructions) with the aim of a
progressive evolution of students’ ways of thinking towards a scientific approach to random phenomena. In order to achieve this aim, natural language is to be fully exploited in its communicative, constructive and reflective functions. For these reasons, the choice of a Vygotskian perspective for the planning of the teaching sequence was strictly related to the nature of the learning aims (in terms of a dialectic relationship to be developed between common concepts and scientific concepts; Vygotsky, 1990, Chapter VI; Douek, 2003) and the role of natural language. Fischbein’s work (Fischbein, Barbat & Minzat, 1971; Fischbein & Gazit, 1984) has shown how relevant schooling is for children’s development of secondary intuitions that may overcome primary intuitions originating from their spontaneous adaptation. In concrete terms, to choose a Vygotskian perspective meant to attribute to the teacher a strong mediating role in the classroom construction of probabilistic knowledge through different kinds of mediation: indirect mediation (by using students’ productions) and direct mediation (by introducing solutions, semiotic tools, voices from the history of probability and/or from books). It meant also to exploit the potential of active imitation (e.g. when students must solve a problem with a method introduced by the teacher, or proposed by a schoolmate, in a similar problem situation) (see Vygotsky, 1978, ch. VI). In the next subsection, we will describe the didactical methodology chosen for our classroom activities and its relationship with the Vygotskian perspective. For further elaboration and examples, see Boero & Douek (2008).

III. 3. The didactical methodology: The teaching cycles

The whole teaching of mathematics and other disciplines in the experimental classes considered in this paper conformed to general educational choices that are typical for teachers engaged in our research team. In the perspective of Research for innovation, the didactical methodology represents both a manner of organizing teaching and an educational environment whose effects on learning are investigated in different mathematical domains. From Grade I on, classroom discussions orchestrated by the teacher (Bartolini Bussi, 1996) follow individual productions and prepare the elements for individual and then collective syntheses of the outcomes of the discussion (Douek, 1999). We will call this a teaching and learning cycle. Exhaustive oral and written wording of students’ thoughts is strongly encouraged by the teacher as a tool to encourage productive discussions.
Teaching and learning cycles

Each teaching and learning cycle is related to a task that should contribute to one of the cultural advances expected in the planning of the teaching sequence. Usually, tasks concern explanatory hypotheses, comparison or motivated choice between alternatives, or motivated prediction of what will happen. After a short presentation of the task, students are asked to produce an individual, extensively-motivated written solution. The teacher supports weak students' work through one-on-one written interaction; when it is needed and possible, weak students have been previously introduced to the task through one-on-one oral interactions outside the classroom. After the individual work, the teacher selects representative answers and provides each student with a copy of the selected answers, and then asks students to compare these with their personal answers. In this phase, it is possible that the teacher needs to introduce an answer not produced by students (as "the answer by a student of another class").

After comparison of answers, a collective discussion orchestrated by the teacher develops. Based on a-priori analysis and (possibly) the activities performed in other classes, the teacher knows some possible approaches to the cultural advance at stake; thus, he/she can profit from students' interventions in order to guide the discussion toward the desired outcome. Remember that the choice and formulation of the task has been made in order to favor students' approach to the cultural advance. Note also how the fact that the teacher is a member of the research team is needed (in the preliminary activities as well as in the first complete experiments) to achieve a relatively quick optimization of the formulation and management of the tasks.

The discussion usually ends with a conclusion reasonably close to the expected outcome. In some cases, indirect mediation by the teacher (i.e. mediation that exploits students' productions) is not sufficient, so that it is necessary to introduce ideas produced in other classrooms or in the culture. The teacher can propose the active imitation of solutions (that have been produced by some students or introduced by the teacher) by asking students to solve a similar problem according to a given model of the solution process. However, as we will see later, the need for direct mediation seems to depend also on the quality of classroom discussion: for some tasks, the more students engage in listening to the others and intervening according to the thread of the discussion, the less the teacher needs to perform a direct mediation.

According to its nature, the conclusion can be followed by some examples, or exercises, or personal formulations of the conclusion.
itself, in order to involve those students that have not constructively taken part in the discussion.

The next step consists in the individual written synthesis of the discussion and its conclusion. For this phase, the weak students are supported by the teacher through written suggestions or questions.

Next, the teacher selects some students' productions and gives one copy of them to each student, in order to start the second discussion, whose aim is to produce a shared synthesis of the work done, which students will report in their copybooks. In this phase, the teacher can introduce technical terms or compare work with official texts. However, in some cases the synthesis represents only a provisional step that will be used as a starting shared point for further work: it can still contain ambiguities, alternative solutions (not necessarily correct!), reasoning shortcuts, etc.

This ideal structure of a learning and teaching cycle can be adapted to different needs: the initial written production of solutions can be replaced by an oral production of some solutions reported on the blackboard, the conclusion of the discussion can be guided by the teacher towards the direct production of a shared text written on the blackboard, or the individual syntheses produced by students can open a second phase of comparison, discussion, conclusion, individual synthesis, etc. For important steps in the sequence, the conclusion of one cycle usually is the starting point of another cycle. One cycle lasts for 4-6 hours during three or four days.

Comparing the ideal structure of our cycles with other structures in the literature, we detect some analogies but also relevant differences with our cycles, depending on the different theoretical assumptions concerning learning. With an eye to Brousseau's Theory of didactical situations (TSD: see Brousseau, 1997) for a comparison, we can see how, in our case, at the beginning of the cycle students are requested to produce solutions that are suitable for verbal communication, in order to push them to integrate thinking and wording (according to the hypothesis that verbal language has a constructive role in thinking processes, and a constitutive role for knowledge - Sfard, 2001; Boero, Douek & Ferrari, 2002). Productions are individual, and they are supported by the mediating role of the teacher, according to Vygotsky's idea of zone of proximal development; no group work is usually planned (with the exception of situations in which it is strictly needed for functional reasons: e.g. in the case of a collection of experimental data, where one student produces the data and another student records them in parallel). The discussion starts from the comparison of some students' productions selected by the teacher, who plays a strong mediating role between students' productions and
the learning goal to be achieved (again according to the hypothesis that the teacher’s role must be exercised in the zone of proximal development, the metaphoric space between students’ autonomous solutions and the solutions that can be achieved through the help of an adult or a more competent peer: Vygotsky, 1978, Chapter VI). The conclusion can be a provisional step (even if it is formulated as an official text reported in students’ copybooks), according to the idea that students’ common concepts must develop gradually towards the scientific concepts progressively introduced by the teacher (Vygotsky, 1990, Chapter VI). According to the terminology of the TSD, for many tasks the devolution (Brousseau, 1997) is only partial: students know that their success in solving a problem may depend on the guide of the teacher, that the teacher can provide solutions for difficult tasks, and that in such a situation their role is only to focus on the problem and understand in what direction to search for a solution (even if they cannot produce the solution), in order to better appropriate the solution offered by the teacher.

In terms of educational goals, this approach clearly demonstrates the fact that students are those whose task is to learn, and the teacher’s main role is not to ensure the conditions for adaptive (or constructive) learning, but rather to provide the students with cultural tools, examples of how to reason in order to solve difficult problems, strategies to be imitated in similar situations (whether produced by schoolmates or offered by the teacher), etc. This approach does not mean that students are not encouraged to produce new ideas – indeed, some of our research developments related to the didactics of the fields of experience (including those presented in this paper) concern the delimitation of the students’ constructive power and the conditions under which it can be effective. What we mean is that a lot of relevant mathematical knowledge must be transmitted to students, and that students must be prepared to grasp, adapt and use it in the problem situations proposed by the teacher.

An important remark concerns the fact that the structure of the didactical cycles is not adopted only in the case of the sequence on probability. It is the main teaching and learning structure for the classes that work in the perspective of the Genoa Team Project for primary school (Douek, 1999, 2003; Boero & Douek, 2008). We add that we have seen how the introduction of the cycle structure for a single didactical sequence on a demanding subject (e.g. probability, or sun shadows, or proportional reasoning) results in a very difficult situation for the teacher and the students, due to the intrinsic difficulty of the subject adding to the difficulties inherent in the break of the didactical contract previously established in the classroom.
III. 4. Preparing, performing and replicating the classroom activities: the role of the teachers

The long-term teaching experiment reported in this paper was prepared by partial experiments performed in two classes (whose teachers were members of our team), which were intended to ascertain whether the chosen didactical situations were suitable to allow students (under the guide of the teacher) to achieve the learning aims related to the identified cultural advances.

Those partial teaching experiments have also been performed in order to find and experiment with suitable technical solutions for the formulation of the tasks and their management in the classroom. Some information and documents about those preliminary experiments are reported in the Genoa Team website (Gazzolo & Massi, 2003).

The teaching experiment involving the whole sequence of activities was performed in the class of another teacher, Teresa Gazzolo, in the years 2000-2003 (18 students coming from a middle-class environment). The whole sequence (without relevant changes: only timing was modified, from Grade I to Grade V in two classes and from Grade II to Grade V in two classes, and some tasks were better formulated) was replicated in four other classes in the years 2003-2007. Their teachers had not taken part in the planning of the 2000-2003 teaching sequence; they had only participated in the a-posteriori analysis of the experiment and discussion of the results, and in the revision of the teaching sequence (thus the revised sequence was a shared outcome of the common work with the teachers engaged in the preparatory activities and in the first complete classroom implementation of the teaching sequence). Since 2005, another experiment involves five classes whose teachers (members of the Genoa Team) only have at their disposal the internal report on the first complete teaching experiment; they can freely adapt the teaching sequence to the reality of their classes and their personal preferences (in this case, timing was changed: in two classes, the sequence was implemented from Grade III to Grade V).

The aims of the teaching experiments (both on the students’ learning side and on the research side) were shared by the two teachers who have been involved in the construction, classroom implementation and analysis of the preliminary activities, by the teacher who joined them during the analysis of their classroom activities and contributed to the construction and classroom implementation of the first version of the whole didactical sequence, and by the four teachers who joined that group afterwards.
The need to involve teachers in the research team is a common character of the Italian research for innovation in mathematics education and corresponds to four goals:

- coherent management of the experimental teaching situations: in the experimental phase, the teacher must open and equip the way through his/her personal classroom implementation of the didactical plan (established according to the \textit{a priori} analysis);
- attention paid to the conceptual nodes identified in the \textit{a priori} analysis, in order to profit from students’ initiatives and productions through real-time choices (and possibly adapt the \textit{a-priori} planning to the classroom reality), and to identify crucial events for field notes (cf. the role of the teacher as participant observer in Eisenhart, 1988);
- conscious, productive participation in the \textit{a-posteriori} analysis of his/her classroom work according to the aims and methods of the research team;
- participation in the elaboration of further solutions for the educational problems encountered during the experiment in the classroom.

We must add that the fact that a teacher who shares research responsibilities in the research team is (according to our experience) more likely to be transparent (as concerns information about his/her didactical style, his/her beliefs about priorities in mathematics teaching, his/her educational values, etc.) than a teacher who hosts an experimental activity without being involved in the research team. This fact is relevant to elaborating hypotheses to explain differences between classes that implement the \textit{same} teaching sequence (see later).

\textbf{III. 5. The collection and preliminary analysis of data}

Collected data concern the five experimental classes that have already taught the whole sequence and consist of: all the individual texts produced by students, audio recordings of all classroom discussions and individual student-teacher interactions concerning probability tasks, and field notes taken by the teachers during and immediately after the activities on probability tasks (in particular, field notes document the expected cultural advances related to the steps of the teaching sequence, and they aid in the analysis of the audio recordings).

Other, less complete, data concern preliminary experiments as well as further work by teachers who had not taken part in the core, complete experiment in the five classes.
In order to organize the analysis of data, we identified as salient episodes those classroom episodes where students (under the tasks designed according to the a-priori analysis), for the first time in the classroom history, produced ideas that (conveniently mirrored by the teacher and discussed under her guidance) represented relevant advances in the probabilistic manner of viewing of the classroom. Relevance of those advances means that they were strictly related to the aims behind the steps selected in the planning of the teaching experiment. Students’ ideas may be individual contributions immediately caught and used by the teacher or contributions derived from social construction, based on ideas brought by two or more students.

In the next two subsections (III.6 and III.7), we will present the criteria followed to evaluate learning results and the teaching sequence, together with some related data. In sections IV and V, we will see how the analysis of the salient episodes allowed us to produce new knowledge about the mechanisms of argumentative construction of concepts and the conditions that could enhance them or prevent them from working.

III. 6. The evaluation of learning results

Concerning evaluation of learning results, the method we chose is consistent with the rationale of a long-term sequence in a given field of experience: the mastery of the field of experience (in terms of competencies shown in tackling the tasks, participating in the discussions and synthesizing their outcomes) is the main aim. It can be evaluated through the following informal but informative evaluation tools:

a) individual solutions for the tasks (particularly the last three tasks of the sequence - see S9, because they have summative character for the aims of the previous part of the teaching sequence);

b) level of the contributions to classroom discussions, particularly in the last discussions (see S8, S9): this information concerns only those students that participated, but is important in order to understand how some students were able to go beyond learned notions and take further autonomous steps;

c) individual written syntheses of discussions: this information concerns all students at the end of each learning cycle; syntheses reveal how the content of the discussions, particularly the new notions reached by students and/or injected by the teacher, have been assimilated. The evaluation is not easy: a positive evaluation depends on the student's thoughtful reconstruction of the discussion and on his/her personal formulation of its conclusions;
d) recordings of individual student-teacher interactions (for those students that had not been able to produce satisfactory individual solutions (a) or written syntheses (c)).

The estimation that about 75% of students achieve sufficient mastery of probability thinking at the last steps of the sequence is a reasonable evaluation of global learning results in the five experimental classes.

Students have been put in the group of sufficient learners when at least two individual tasks out of the last three were solved in a correct way and at least two individual written syntheses out of the last three were sufficient (meaning the main steps of the discussion and the conclusions were thoughtfully reported with appropriate personal wording). Also, about 35% of all students were able to produce useful contributions for the advancement of at least one of the last three discussions.

75% of students achieving sufficient learning is an average estimation that derives from different results in the five classes, mostly depending on the socio-cultural environment: in the worse environmental situation, about 60% of students achieved sufficient learning results, versus 85% in the two best environmental situations. However, we cannot tell whether the differences in results depend on the environmental difficult situation and/or the educational choices of the teacher (the more interventionist of the team); see later for further details and discussion.

Approximately one half of the students that had not been autonomous in dealing successfully with the tasks of individual solution and synthesis did achieve success when supported by the teacher in one-on-one interactions.

III. 7. The analysis and evaluation of the teaching sequence

The analysis of the teaching sequence has been made according to three different levels of detail (with different related tools and criteria):

- Analysis of the whole sequence, in terms of: timing of the main steps of the common planning (and comparison between different timings in different classes); and long-term learning results (final evaluation and inherent tools, and comparison between different classes). Concerning timing of the whole sequence, we have experienced ways to flexibly adapt it. Until recently, the sequence of the first complete teaching experiment (from Grade I to Grade IV) was also presented as a longer five-year sequence (from Grade I to Grade V) and as a four-year sequence delayed from Grade II to Grade V. It is currently being tried from Grade III to Grade V in two classes.
It is interesting to observe that, while the first two steps seem to be superfluous when students enter the sequence in Grade III, all the other steps represent a challenge for students, regardless of when they meet the related tasks. However, this is not an unexpected outcome (we have some evidence that even university students have difficulties with tasks related to the other steps). Learning results (see previous subsection) do not seem to depend on the location of the whole sequence (from Grade I to Grade IV or from Grade II to Grade V or from Grade I to Grade V). The fact that the five classes that have completed the experiment have invested approximately the same time (around 90 hours) in the whole sequence (whether distributed over four years or five years) does not allow us to evaluate if more time (or less time) could influence learning results.

- **Analysis of the didactical situations related to a specific task** in terms of the nature (and timing) of the different kinds of classroom activities (explanations, classroom discussions and the role of the teacher during them, students' individual work) in each class, with comparison between different classes and within the same class for different situations.

We found no evidence of relevant differences concerning the difficulties met by the four teachers who have tried the improved sequence, though we observed different styles of classroom management, with different distributions of time among the different kinds of activities and different roles played by the teachers according to their beliefs (recall that the fact that the teachers are members of the research team allows to establish correlations between their explicit positions during the team meetings and their classroom choices). In particular, the more interventionist teacher (she whose interventions are more frequent and directive during classroom discussions, and whose explanations take longer and carry heavier weight in classroom activities) is she who says that she does not expect much advancement of knowledge from peer interaction, and she tends to exploit peer interaction mainly to involve students in the proposed activities. It is interesting to note that she comes from many years of teaching in very difficult social environments (and she recognizes how she had to modify her initially less-interventionist position in order to face the difficulties met in those classes).

- **Analysis of salient episodes** (see III. 5 for a definition). The analysis has been performed in terms of the role of the teacher in those episodes, their occurrence in different classes, the potential of social interaction, the role of argumentation and communication in the classroom, and the role of the values shared in the classroom. Specific
indicators and reference models (partly innovative) for the analyses will be proposed in the next section.

IV. BASIC RESEARCH DEVELOPMENTS: ON THE CONSTRUCTIVE FUNCTIONS OF NATURAL LANGUAGE IN SOCIAL INTERACTION

In accordance with the features of Research for innovation, we now try to show how the didactics of fields of experience (presented and exemplified in the previous section) can work as a long-term educational context for local in-depth analyses and research developments concerning the main working hypothesis (argumentative approach to probabilistic thinking), which underlies the planning and experimentation of the teaching sequence.

The aim of this section is to analyse some mechanisms through which argumentation in social interaction resulted in knowledge construction in the teaching experiment concerning random events and probability from grade I to grade IV. The analyses we performed demonstrate a peculiar function of natural language in classroom discussions: as a tool to transform the content of the discourse through interactive mechanisms of linguistic expansion based on key expressions.

IV. 1. THEORETICAL PERSPECTIVE

In the last decades, an important trend of research in mathematics education has been the increasing attention paid to language and semiotic aspects in the construction of mathematical knowledge, both in an individual and in a social construction perspective. This occurred together with research advances in other domains (psychology, linguistics, hermeneutics). Let us consider the perspective of the constitutive character of natural language (Bruner, 1986, Chap.4): on the one hand, it suggests considering whether other semiotic systems (in particular, algebraic language) share the same potential, and how students can approach the mathematical realities inherent in the specific expressions of those systems (Sfard, 1997, 2000; Radford, 2003); on the other hand, it opened the way to study how the mathematical realities are constituted during verbal activities in the classroom (Sfard, 2001).

With reference to these branches of research, we had already produced some research results that constitute the starting points for the research reported in this paper. Boero (2001) and Consogno (2006) consider how mathematicians deal with algebraic or natural language written expressions. In the case of algebraic expressions
(Boero, 2001), crucial steps of a mathematician’s activity consist of reading the algebraic expressions produced by him/her: sometimes, this reading suggests ideas that go far beyond what the reader considered during the writing phase. The novelty can consist of the discovery of a way to simplify the expression, the discovery of a new meaning, or the anticipation of some moves that will achieve the goal of the activity. In the case of natural language expressions, Consogno (2006) considers the flow of the writing/reading phases during individual activities of conjecturing and proving performed by undergraduate mathematics students. The Semantic-Transformational Function (STF) of natural language is the construct that accounts for some advances of students’ conjecturing and proving process. The student produces a written text with an intention he/she is aware of; then he/she reads what he/she has produced. His/her interpretation (suggested by key expressions of the written text) can result in a linguistic expansion and a transformation of the content of the text that allow advances in the conjecturing and proving process.

Douek (1999) is concerned with the analysis of the role of argumentation during classroom discussions aimed at the construction of mathematical concepts in activities of elementary mathematical modelling of physical phenomena. She identifies lines of argumentation whose development and crossing contribute to the enrichment of concepts in terms of reference situations, operational invariants, linguistic representations (according to Vergnaud’s definition of concept: Vergnaud, 1990), and in terms of maturation towards the level of scientific concepts (Vygotskij, 1990, Chapter VI). Her analyses show how, in some cases, a line of argumentation develops through someone’s interpretation of linguistic expressions produced by others, far beyond their intention in producing them.

The aim of the study reported in this paper is to see if the STF of natural language (Consogno, 2006) can account for the development of a line of argumentation during a classroom discussion (by focusing on the phases when oral productions by some students are interpreted by other students), and to characterize how it works. This section addresses two questions:

A) Can classroom social construction of mathematical meaning be interpreted in terms of STF (i.e. of semantic transformations that happen through linguistic expansions produced by someone that are suggested by key expressions uttered by some others)?

B) Can a student profit, in classroom discussion, from others’ interventions (in order to develop his/her intuitions) through mechanisms that involve the linguistic transformations of his/her own expressions?
IV. 2. Methodology
Some salient episodes (see III. 5, last paragraph, for a characterisation of them) will be analysed according to the STF hypothesis: we will try to detect possible links between verbal expressions produced by someone and the development of the content of the discussion performed by some other(s) by expanding those expressions and transforming their meaning. In the last subsection, we will discuss potential limitations of this kind of analysis, as well as some possible developments.

IV. 3. Some salient episodes
We have chosen salient episodes that exhibit different ways of advancement of knowledge through argumentative social interaction; they happened in the same class (the first class that implemented the whole sequence); they are representative of a set of 7 salient episodes recorded in that class. As we will see in the next subsection, the same ways of advancing knowledge have also been identified in the other experimental classes, though with a different distribution among the classes (see V.2). The reported episodes are salient because they allowed the teacher to achieve the aims of 3 steps of the teaching sequence (see III.1) by exploiting the students' advancements of knowledge.

Episode 1 (see Step 5, first task)
Grade III: students approach the idea of ratio between the number of favourable cases and the number of all cases as a measure of probability of an event. Students are asked to make a choice between two games: the game with a coin (by betting on heads or tails), or the game with a dice (by betting on one of its digits). One student, Giulia, writes:

In the case of the coin, there are many more possibilities. For instance, suppose that in two labyrinths there are 2 paths (in the former) and 6 paths (in the latter). The 2-paths labyrinth offers more possibilities to get out, if in each labyrinth there is only one exit.

The text produced by Giulia is chosen by the teacher to feed a classroom discussion, because it can help the students to compare on the same, neutral ground (labyrinths) two different random situations. Note that in the text produced by Giulia (as well as in all the other texts) there is no trace of reasoning in terms of ratio (“more possibilities” concerns only the comparison of 6 against 2). Recall that the notion of ratio has not yet been dealt with in this class. Note also that Giulia orients the discussion towards a simplified, yet abstract, model of labyrinth. The teacher asks to students to take a
position on Giulia's text and to evaluate if her last sentence ("The two paths labyrinth... ") was necessary or could have been omitted.

Anna:  I agree with Giulia that in the 2-paths labyrinth you get out earlier, in the case of the 6-paths labyrinth you must try all the paths and you spend a lot of time.

Matteo:  But in the 6-ways labyrinth you do not need to try all the paths, because for instance the first time you fail the exit, but then at the second or third trial you may find the good way to escape... You don't need to try all the paths!

Giovanni:  It is necessary to consider the condition posed by Giulia, namely that there is only one exit, otherwise all the paths might have an exit, and it would not be a labyrinth any more!

Mattia:  If a labyrinth had more exits than paths with no exit, practically it would be very easy to escape; on the contrary, if the labyrinth has the same number of paths and exits, ... it would be easier but the exits must be more than one half of the number of the paths.

Some voices: less than one half!

Teacher:  I would like Mattia to repeat his sentence - please, listen to him, then we will discuss what he said.

Mattia:  Can I make an example? In the 2-paths labyrinth there is one exit, while in the 6-paths labyrinth there are 3 exits; in order to make the 6-paths labyrinth easier than the 2-path labyrinth, you must put exits to more than one half paths, because if in the other labyrinth there are two paths and one exit, it is one half.

Anna's and Matteo's considerations suggest to Giovanni the reason why the condition posed by Giulia is necessary: the expression “all the paths” (be it necessary to try all of them, or not) can suggest the fact that if “all the paths have an exit” then it is sure that one can escape from the labyrinth in each trial. The situation is transformed by passage to a non-labyrinth limit situation. Then the extreme cases of one exit and six exits opens the way to Mattia to consider the number of exits in the 6-paths labyrinth as a variable that can take values between one and six. He tries to express the idea that the right comparison with the 2-paths situation must be made by considering “one half of the number of the paths” as the discriminating case. In terms of the STF, he performs some linguistic expansions (“more exits than paths with no exit (…) the same number of paths and exits” in his first intervention, and then “an exit to more than one half paths” in his second intervention) of the limit situations uttered by his schoolmates, which results in a transformation of the situation: the number of exits becomes a variable related to the number of paths.

In this way, Mattia moves from the set of cases proposed by his schoolmates to a general consideration of the relationships between
the number of exits and the number of paths. From Mattia's second intervention on, several progressively more precise interventions will concern “3 exits out of 6”, “one half of the exits”, and so on, until reaching the explicit comparison between 3 out of 6 and 1 out of 2 as “one half” in both cases.

**Episode 2 (see Step 6, first task)**

Grade III: in couples, students will throw two dice; they will bet on odd or even according to the number got by adding the digits of the dice. Before playing the game, the question is: “Is it better to bet on odd or even?”

Individual answers follow, and a discussion takes place. At the beginning of the discussion, students consider odd outcomes: 3, 5, 7, 9, 11; and even outcomes: 2, 4, 6, 8, 10, 12. Even seems more likely to come out because the number of even outcomes is bigger. But…

- Elisa: I agree with Mattia, as he considers the results.
- Giulia: Mattia has considered all possibilities, because he has considered the two dice and has put the results and (I think) has looked at all possibilities
- Teacher: Is it the same thing to think of the result or to think of the two dice?
- Mattia: It is the same thing … no… yes!
- Giulia: If you think of dice… to the digit shown by your dice… because the result is one digit plus another digit that makes a result. Before adding them, those two numbers are alone, they are not together… because if one casts 3 and the other 4
- Roberto: For instance, 4 is a number and 3 is another number, as Giulia said, if you add them, they make 7, but before putting them together, 4 is a solitary number and 3 is another solitary number, then when they go together we get a number formed by smaller numbers
- Giulia: Yes, but before getting the result, the two numbers can be other numbers.

The teacher asks Giulia to make an example, and then she invites the other students to produce other combinations. The way is open to consider all possible equally likely outcomes. Note that, in Italian, the digits of the dice are called numbers.

In the reported fragment, Giulia re-elaborates the distinction (suggested by the teacher) between the dice and the result in terms of numbers: the addends and the results. The intervention of Roberto not only echoes Giulia's intervention, but expands it and suggests a transformation of the content (offering as evidence, by saying "for instance", the fact that the couple 3 and 4 is an example; and the fact that the sum is a "number formed by smaller numbers"). Note how
the syntactic construction "a number formed by smaller numbers" opens the possibility to see a number as formed by smaller numbers in different ways. From this, Giulia is able to see how those "smaller numbers" can be different from 3 and 4. We can interpret "a number formed by smaller numbers" as the key expression suggesting a linguistic expansion that results in a semantic transformation of Giulia's original idea. The interpretation of the situation by Roberto comes back to Giulia as an opportunity to enrich her way of thinking and contribute to the advancement of classroom discourse.

**Episode 3 (see the task of Step 8)**

Grade IV: students compare throwing of two dice with the morra game in order to better understand what a random event is. This occasion came from the comparison of the histograms of their outcomes after a relatively high number of trials (120 for each game). Students start the discussion by considering some exterior features of the histograms, then they realise that the comparison must be made by considering the range 0-10 in the case of the morra game and 2-12 in the case of the dice. A strange fact captures students' attention: in the case of the dice, the outcomes 2 and 12 are the least frequent, while in the case of the morra game 10 is one of the most frequent outcomes, and the frequencies of 0, 1 and 2 are very low.

Elisa: I think that the last number of the morra became high because you decide what number to cast, and, for instance, if you cast 5, probably also the other casts 5.

Emanuele: Yes, probably he also casts 5, because sometimes it happens that it is easy to cast 5.

Voices: Yes, it is easier!

Giulia: 5 is very easy, because you must cast your number quickly, and so you open your whole hand and then 5 is easy.

Matteo: With the dice, 12 is less frequent because it has only one possibility, and the same for 10 in the morra, but 10 is more comfortable, on the contrary with dice it is not easy that 6 and 6 happen frequently.

Marco: I think that in the morra game 10 was frequent because when you cast your hand it is difficult that you cast the number you have thought, because you must do it quickly.

Some reactions: But you do not know what number was cast by the other?

Danilo: Marco does not mean the sum of the two hands, but the number he is thinking of, he was thinking to cast 2 and instead he cast 5 because 5 is easy to cast.

Pietro: Because when you make 5 you do open your whole hand.
Evandro: Two is difficult as well as 1, because if you play quickly you must close 4 fingers and leave 1.

At the beginning, Elisa explains how to get 10; her utterance offers Emanuele the possibility of an interpretation (i.e., a linguistic expansion that transforms the meaning of what Elisa told) in term of easiness to cast 5 (seconded by others as well). Giulia adds an explanation in terms of body mechanisms (the easiness expands in her words into the representation of a physical action probably suggested by the key word “easy”). Matteo comes back to the comparison between 10 (morra) and 12 (dice) in terms of random/not-random events. To say “10 is more comfortable” is an interpretation of preceding comments that expands into a more complex discourse, a transformed situation involving the comparison with the purely random outcome 12. The next part of the discussion leads to a comparison between the outcomes 10, on one side, and 1 and 2, on the other, in the morra game. Marco’s intervention, appropriately interpreted by Danilo (in psychological terms) and Pietro (in physical terms), seems to offer Evandro the opportunity to shift to the cases of 1 and 2. In this case, the transformation realised by the linguistic expansion based on the key expression “open your whole hand” occurs as an alternative to the situation represented by Danilo’s and Pietro’s utterances.

IV. 4. Discussion

The analyses of salient episodes seem to provide us with empirical evidence for the hypothesis that the STF of natural language can play a crucial role in social construction of knowledge through linguistic expansions based on key expressions, which result in a transformation of the content of the discussion. However, we must take some distance from such a conclusion. Collected verbal data need an interpretation by the researcher, and this interpretation can be subjective. In the second episode, Giulia could have continued to think about her intuition independently from Roberto’s utterances, and finally she could have realized that “the two numbers can be other numbers.” One way to escape subjectivity in interpretation might have been asking Giulia where the idea of “other numbers” came from. It would have been necessary to ask this question immediately (in order to avoid an arbitrary reconstruction based on the subsequent steps of the discussion), but the effect might have been to disturb the flow of the discussion. The main aim of a teacher in classroom situations is different from the aim of a researcher in laboratory situations! We think that another way to reduce subjectivity might be to collect a lot of episodes of advances in classroom discussions and check the
frequency of those cases in which advances can be interpreted in terms of STF. If the frequency is high, the plausibility of the hypothesis is reinforced. For instance, in the case of our first complete teaching experiment, we have identified 9 advances that occurred in classroom discussions through students' contributions, 7 of which can be interpreted in terms of the STF of natural language (three have been reported in the previous subsection).

Let us suppose now that, in the reported episodes, things worked according to the STF hypothesis. An interesting issue related to research questions A) and B) concerns the roles of the others' interventions in the discussion. While, in the case of the interpretation of one's own written production, the unique role of the written text is that of an external reference for an inner evolution of thought, in the case of social interaction, the others' interventions play different functions. The voice of a student can provoke (through specific key expressions) an interpretation by another student related to his/her perception of the evoked situation, which comes back to the first student as an enrichment or a transformation of his/her original intuition (like in Episode 2). In that case, the other student plays a role that could be made internal through a mechanism of inner dialogue supported by a written text (like in the episodes analysed by Consogno, 2006).

In other situations, a chain of development happens: different students can play complementary roles to transform the situation under consideration. It may happen that the focus moves from one situation to the opposite situation (like in Episode 3); or that two (or more) complementary interventions open the way to the consideration of the whole range of possibilities between those evoked (like in Episode 1). In the last case, social construction of knowledge seems to reveal its highest potential. According to these considerations, focussing on the STF of natural language seems to offer the researcher the possibility of classifying different patterns of social construction of knowledge in terms of different mechanisms of linguistic expansion. This suggests a need to characterise the variety of mechanisms that are of interest in the perspective of the STF of natural language.
V. FURTHER BASIC RESEARCH DEVELOPMENTS: SOME MECHANISMS OF THE STF OF NATURAL LANGUAGE IN SOCIAL INTERACTION

Previous analyses had demonstrated a peculiar function of natural language in classroom discussions: as a tool to transform and develop the content of the discourse through interactive mechanisms of linguistic expansion based on key expressions.

Taking into account the analyses reported in the previous section (see also Consogno, Boero and Gazzolo, 2006), in the next subsection we will propose descriptions of the three kinds of mechanisms that emerged in the previous case study. The descriptions will exhibit some features that allow us to recognize those mechanisms in classroom social interactions and their functions in the development of classroom discourse.

In the following subsection, we will consider some data concerning the differences between the five experimental classes of our teaching experiment on the approach to probabilistic thinking, and we will discuss possible interpretations of those differences.

The last step will be to analyse some salient episodes in which the proposed mechanisms have played a crucial role in the development of classroom discussions in another teaching experiment concerning the approach to mathematical proof in twelve classes of Grade VI. This teaching experiment was also performed by teachers belonging to the Genoa Mathematics Education Research Team. The aim of this subsection will be to show how the three kinds of mechanisms can be detected in very different tasks and mathematical domains, thus showing their character as general mechanisms in the social construction of mathematical knowledge.

The analyses of the salient episodes will have a dual aim: to provide experimental evidence for the relevance of the three mechanisms in the social construction of knowledge and in the development of mathematical reasoning, and to show how their functioning can be interpreted in terms of the STF of natural language. The analysis of the salient episodes will be performed according to a modelling perspective: students' utterances will be interpreted as if their thinking processes would fit the models of reasoning proposed by us. This is a legitimate choice until students' words explicitly contradict our interpretation. Naturally, different and equally coherent interpretations might be possible in some cases.
V. 1. Three kinds of mechanisms of development of classroom discourse

I. Evolution of a personal interpretation of the situation (like in Episode 2)
One student's interpretation of the problem situation is enriched by and integrated with the interventions of some schoolmates who propose other interpretation(s) of the same situation, up to the full apprehension by the first student and his/her relevant contribution to the solution of the problem in the classroom discussion.

II. From a situation to the opposite one, to a wider perspective (like in Episode 3)
Students' contributions describe two opposite situations related to the task (for instance, one case fits the conditions of the task, while the other escapes them). This contributes to construction of knowledge by offering a wider perspective for a discourse that embraces both cases and allows an advance in conceptual construction and reasoning.

III. From single cases to generalisation (like in Episode 1)
Students propose some similar cases related to the task, and then engage in a collective process of induction by considering common features of the evoked cases. A general statement is the outcome of this process.

V. 2. Some data on differences between classes, and possible interpretations of them
We found relevant differences between the five experimental classes, which concern the occurrence of these mechanisms: with the same 22 tasks, the whole sequence can take from a minimum of two salient episodes up to a maximum of 14 salient episodes. A possible interpretation concerns dependence on the educational context. Indeed, in the extreme cases we know that teachers consciously adopt different styles for managing classroom activities (particularly classroom discussions). As was demonstrated in a previous section of this paper, the extensive knowledge of teachers' attitudes, beliefs and educational values can offer elements to interpret relevant differences. For instance, in the class where only two episodes were detected, the teacher is the most sceptical person on the team regarding the productivity of peer interaction. The fact that teachers teach the same class for five years can amplify differences that reflect teachers' educational values and beliefs.
We have also found relevant differences between the five classes as concerns the relative frequency of the three types of mechanisms: in particular, in one class the distribution of ten episodes among the three types of mechanisms was (1, 7, 2), while in another class the distribution of twelve episodes was (5, 2, 5); here, differences might depend not only on the teacher's choices (in particular, on which kind of interaction is promoted by the teacher: cooperative or competitive), but also on the personal characteristics and mutual relationships of leading students in the classroom and on the social values shared in the classroom.

In general, knowing the teachers' way of managing classroom situations justifies the hypothesis that the didactical contract (see Brousseau, 1997) is one of the elements that can explain the differences. However, other elements (related to affective aspects and the socio-cultural environment) seem to be influential as well. In particular, the present widening of the experimentation to classes coming from very different social environments offers some preliminary data that show the importance of rules of interaction that are learned out of school as social norms belonging to the social environment (cf. Bernstein, 1996). This is not a strange hypothesis to explain the differences between the classes: if we look at the three kinds of mechanisms, we find that (for instance) the second mechanism strongly depends on the legitimacy of opposing others' opinions, while the first mechanism needs a collaborative style of interaction - at least between the protagonists of the interaction.

We can ask ourselves whether there are common values and behaviors that students must have in order to activate the three mechanisms, in particular values and behaviors that can be enhanced by the teacher. We think that listening to others, freely using (and transforming or contradicting) schoolmates' productions, and sharing the aim of solving the problem situation as a collective enterprise are three necessary conditions strictly related to the nature of the three mechanisms. Those conditions do not seem out of the reach of a teacher (or a team of teachers) who is given sufficient time with students, even if the educational task is more demanding in some environments than in others. The didactical methodology described in a previous section seems to be suitable to promote those behaviours that we are considering now: the systematic use and comparison of students' productions educates students to take seriously and try to understand and use what their schoolmates say, while the whole teaching style (including the guidance of classroom discussions) is aimed at students' sharing of the goals of the activity.
V. 3. The same mechanisms in other mathematical performances

The examples that follow have been taken from transcripts concerning the following task, which was given to twelve classes of junior high school (Grade VI in Italy) at the end of the school year 2005-06, with the goals of developing mathematical reasoning and approaching mathematical proof:

Prove, in general, that two consecutive numbers have only 1 as their common divisor

The educational aim of the task was to offer an occasion to move from justification based on examples to a general argument concerning whatever numbers. An empirical search for divisors of consecutive numbers soon becomes heavy and boring; thus, reasoning in general can become (under the teacher's guide) a shared opportunity in the classroom.

In the a priori analysis of the task, we had considered two possible strategies (one based on the consideration of remainders, the other based on the consideration of the distance between two consecutive multiples of the same number). It was expected that the possibility of different strategies arising might offer an opportunity to compare and share different ways of reasoning to solve the problem.

The research aim of the task was to analyse different ways knowledge can be socially constructed. Indeed, the variety of possible strategies, the shared need for general arguments and the complexity of linguistic and mathematical aspects inherent in the task offered an opportunity to observe how different, personal verbal contributions (and ways of thinking) might "converge" in the social construction of a solution. For instance, we can say that a number is a divisor of another number if it divides it or if the remainder of the division of the latter number by the former one is zero. These different ways of speaking about the divisibility of one number by another correspond to different ways of thinking about that concept, thus offering students different opportunities to approach the problem.

Mechanism 1

Paolo: A number is a divisor of another number… it means that it divides it… A number divides another number when it is contained exactly a certain number of times in it, nothing remains (non resta niente, in Italian) in the dividend. Now I have two consecutive numbers… A number and the following number… Nothing remains in the previous number… (long silence)

Lucia: A number is divisible by another number when the remainder is zero (il resto è zero, in Italian). If I move to the following number… the remainder… (long silence)
Paolo: The following number is the previous number increased by one... Thus the remainder is one... If I divide the following number by a divisor of the previous number, I get one as remainder, so the following number is not divisible by that divisor.

In this case, the verb remains (resta in Italian) uttered by Paolo suggests the noun remainder (resto in Italian) to Lucia, while the expression “the following number... the remainder” uttered by Lucia suggests to Paolo both the “increased by one” and “the remainder one” (a crucial linguistic expansion in order to get a full understanding of the problem situation). This allows Paolo to conclude his reasoning by considering the remainders (zero, i.e. divisibility; one, i.e. no divisibility) of the division of two consecutive numbers by a divisor of the first number. Note that the transition from the verb remains to the noun remainder (i.e. from an inclusion to a division point of view) performed by Lucia allows Paolo to enter the more familiar situation of remainders of divisions, which students were widely accustomed to from previous grades.

Mechanism II

Rosy: In case of divisibility, the remainder is zero
Lorena: While in case of no divisibility, the remainder cannot be zero
Daniele: In the case of two consecutive numbers... (long silence)
Francesca: In the case of one number and the following one... (long silence)
Ivan: In the case of the following number, we move from remainder zero to remainder one, so the following number is not divisible by that divisor

“In case of... the remainder” is the key expression that allows moving from a situation to the opposite one, and then to a linguistic expansion that embraces both cases and allows the reasoning to be finalized. Note also how Francesca contributes to the debate by transforming the expression “Two consecutive numbers” (coming from the task) into the expression “one number and the following one”, which allows Ivan to see the transition from “remainder zero” to “remainder one”.

Mechanism III

Maria: In the case of two as divisor, we need to move from one even number to the following one, two steps away.
Barbara: While in the case of three as divisor, we need to move from a divisible number to the next number divisible by three... three steps away
Francesco: And in the case of four, four steps away!
Lorena: The distance is growing more and more, when the divisor increases… the distance is the divisor! … (long silence)
Roberto: So if the distance is one, the only divisor is 1.

The expression “… steps away” (“… passi distante” in Italian) allows students to move from one example to another, then the idea of “distance” (“distanza” in Italian) allows them to embrace all the examples in a general statement that Roberto can particularise to the case of interest for the problem situation. Note that, in the Italian language, students can move easily from the adjective “distante” to the noun “distanza”.

**Compound processes**

In some cases, we have observed a different kind of social construction of mathematical reasoning, like in this example, where a process of type III contributes to a process of type I:

Elena: One number and the following one… an even number is followed by an odd number, this means that 2 cannot be a common divisor… It would be a common divisor for the following one, four … I must make a jump… (long silence)
Fabio: It is necessary to make a jump of two places
Stefania: If one number is divisible by three, the following number that is divisible by three is three places away…
Gina: And four places away in the case of a number divisible by four…
Elena: I understand: if one number is divisible by another number, then the following case of divisibility will be as far away as the divisor!

Elena considers even/odd numbers, and (if alone) she probably would not have been able to leave that situation. Fabio sees the jump of two positions, and Stefania and Gina suggest further examples that expand the range of exploration. Finally, Elena realizes that “two places away”, “three places away”, and “four places away” can generalise to “as far away as the divisor”.

In this complex social construction, we see how the expression “make a jump” uttered by Elena suggests to Fabio the expansion “make a jump of two places”, a new interpretation of the same fact evoked by Elena. “A jump of two places” suggests to Stefania the linguistic transformation “Three places away”, which allows Gina to produce another example: “four places away”. Elena integrates those contributions in a more general statement that links back to her initial interpretation of the situation.
VI. CONCLUSION

A long-term teaching experiment (four years, from grade I to grade IV) has been planned and performed in one class of 18 students by a teacher who is a member of our research team, and replicated in four other classes. The teaching experiment involves a sequence of tasks and related activities, aimed at approaching probabilistic thinking according to a field of experience perspective. The results show that most primary school students can learn to deal with random phenomena both in terms of a mathematical treatment, and in cultural terms (by becoming aware of what a random phenomenon is and why some spontaneous or common ways of thinking it are incorrect).

The peculiar characteristics of the experimental situation (all the teachers are engaged in the research team, even if their engagement in the teaching experiment was different - see III.4; all the teachers taught in the same class for the whole period of the teaching experiment - from 3 to 5 years) prevent from drawing conclusions that concern the transferability of the teaching sequence and its learning results to other situations. However, some features of the salient episodes analysed suggest research developments (see below) that could help identify the educational conditions for successful transfer.

The teaching experiment was based on the hypothesis that classroom argumentative activities orchestrated by the teacher could result in substantial advancements in students’ knowledge. Collected data offered the opportunity to understand how the semantic-transformational function of natural language (demonstrated in our previous studies) can play a crucial role in the social construction of knowledge. Three mechanisms have been demonstrated for the STF: they correspond to three different ways peer interaction can function, and we have detected them not only during the long-term teaching experiment reported in this paper, but also in another teaching experiment on a very different subject (the approach to conjecturing and proving in Grade VI).

The analyses of some salient episodes belonging to different teaching experiments show how the three mechanisms of social development of classroom discourse can fit (as descriptive models) what happened in the classrooms, and how the STF model can account for the functioning of those mechanisms (as an interpretative model).

Further directions of research are suggested by the performed analyses: to identify other mechanisms (if any) for social development of classroom discourse, and to investigate the educational conditions (didactical contract, shared values in the classroom, etc.) that allow the mechanisms described in this paper to work. In particular,
listening to others, freely using (and transforming) schoolmates’ productions, and sharing the aim of solving the problem situation as a collective enterprise seem to be three necessary conditions inherent in the nature of the mechanisms, and these are within the reach of the teachers through suitable long-term educational choices. The differences in the activation of the STF mechanisms between different classes suggest further investigations of the conditions under which those mechanisms are activated. At present, we recommend considering a complex interplay between environmental conditions depending on socio-cultural factors, teachers’ beliefs (that could be partly shaped by environmental constraints) and the role of the leading students’ personalities and relations in the classroom.

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